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LETTER TO THE EDITOR

A simple proof of the positivity of the Bondi mass

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Abstract. We show that the Bondi mass of a physically reasonable isolated system is positive.

In two recent papers (Ludvigsen and Vickers 1981, Israel and Nester 1981) spinor techniques similar to those used by Witten (1981) in his proof of the positive mass theorem at space-like infinity were used to argue that the Bondi mass of a physically reasonable isolated system must be positive. These papers, however, fall short of giving a full proof of the positivity of Bondi mass, in that they assume the existence of an asymptotically constant spinor field which satisfies the Witten equation on an asymptotically null, space-like hypersurface. Even though there are good reasons for believing that such fields exist, the standard existence theorems for elliptic partial differential equations are not readily applicable in this case and it is therefore difficult to give a rigorous existence proof.

In this Letter we shall avoid this difficulty by using a non-singular hypersurface Σ which is space-like on some compact region $\mathscr{L}$ with boundary S_0 outside of which it becomes an outgoing null hypersurface $\mathscr{N}$ which meets future null infinity $\mathscr{I}^+$ in a cut S_∞ (see figure 1). By using such a hypersurface, rather than one which is only asymptotically null, we are able to prove the existence of spinor fields which yield a positive expression for the Bondi mass by applying only standard existence theorems based on compact hypersurfaces. This allows us to prove the following theorem.

Theorem. Let $\mathscr{M}$ be an asymptotically flat space–time which satisfies the dominant energy condition and which admits a hypersurface of the above type. Then the Bondi momentum P_a with respect to the cut S_∞ is future pointing.

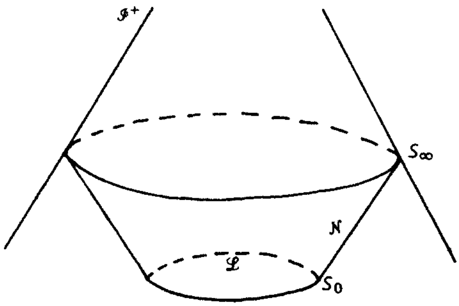

Figure 1. The hypersurface Σ consists of two parts: $\mathscr{L}$, a compact space-like hypersurface and $\mathscr{N}$, a null hypersurface which meets $\mathscr{I}^+$ at S_∞.

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In proving this theorem we shall use techniques similar to those used in our earlier paper (Ludvigsen and Vickers 1981), except that we use a different spinor propagation law on the null surface $\mathscr{N}$. Horowitz and Perry (1981) have independently given a proof of a similar theorem based on an asymptotically null hypersurface. Their proof, however, relies on difficult and recently discovered existence theorems for partial differential equations on non-compact surfaces (Choquet-Bruhat and Christodoulou (1981).

Throughout this Letter we use the abstract two-component spinor notation of Penrose (1968) and the Geroch, Held, Penrose (GHP) spin-coefficient formalism (1973).

We begin by considering spinor fields on the null surface $\mathscr{N}$. Let $(r, \zeta, \bar{\zeta})$ be a Bondi-type coordinate system on $\mathscr{N}$ such that $r = r_0 =$ constant on S_0 and let (o_A, ι_A) be the corresponding GHP spinor dyad on $\mathscr{N}$ such that

$$o_A\iota^A = 1, \qquad o^A o^{A'} = l^a = \partial x^a/\partial r, \qquad l^a\nabla_a o_A = 0,$$

and where $n^a = \iota^A\iota^{A'}$ is chosen such that it lies in the ingoing null hypersurfaces orthogonal to the $r =$ constant cuts of $\mathscr{N}$. A spinor field λ_A on $\mathscr{N}$ will be said to be asymptotically constant if its components $\lambda_0 = \lambda_A o^A$ and $\lambda_1 = \lambda_A\iota^A$ satisfy

$$\begin{aligned} \lambda_0 &= \overset{0}{\lambda}_0(\zeta, \bar{\zeta}) + \mathrm{O}(r^{-1}), \\ \lambda_1 &= \overset{0}{\lambda}_1(\zeta, \bar{\zeta}) + \mathrm{O}(r^{-1}), \end{aligned} \tag{1}$$

where

$$\eth_0\overset{0}{\lambda}_0 = 0 \tag{2}$$

and

$$\bar{\eth}_0\overset{0}{\lambda}_0 + \overset{0}{\lambda}_1 = 0. \tag{3}$$

The differential operators $\eth_0$ and $\bar{\eth}_0$ are the standard Newman–Penrose (NP) (1966) 'edth' operators based on a unit sphere. In terms of a spin frame $X^A_{\boldsymbol{A}}$ ($\boldsymbol{A} = 0, 1$), which is itself asymptotically constant, the components of such a field have a well defined asymptotic limit

$$\overset{0}{\lambda}_{\boldsymbol{A}} = \lim_{r\to\infty} \lambda_{\boldsymbol{A}} = \lim_{r\to\infty} X^A_{\boldsymbol{A}}\lambda_A \tag{4}$$

where $\overset{0}{\lambda}_{\boldsymbol{A}}$ is independent of ζ.

As we showed in our previous paper (Ludvigsen and Vickers (1981), the Bondi momentum $P^{\boldsymbol{AA'}}$ with respect to S_∞ can be expressed in terms of such asymptotically constant fields according to

$$P^{\boldsymbol{AA'}}\overset{0}{\lambda}_{\boldsymbol{A}}\overset{0}{\lambda}_{\boldsymbol{A'}} = \lim_{r\to\infty} I(r) \tag{5}$$

where

$$I(r) = -\tfrac{1}{2}\oint \phi_{AB}o^A\iota^B \,\mathrm{d}\Omega + \mathrm{cc} \tag{6}$$

and

$$\phi_{AB} = \lambda_{C'}\nabla^{C'}_{(A}\lambda_{B)} - \lambda_{(A}\nabla^{C'}_{B)}\lambda_{C'}. \tag{7}$$

In equation (6), $\mathrm{d}\Omega$ is the surface element of the $r =$ constant cuts of $\mathscr{N}$ and cc stands for complex conjugate. An important property of $I(r)$, which can be checked by expressing it in spin coefficient form, is that it only depends on derivatives of λ_0 and λ_1 which are intrinsic to the $r=$constant cut; $I(r)$ is thus completely determined by giving the components λ_0 and λ_1 on $r=$constant.

Our proof is based on two main steps. Firstly, we show that there exist asymptotically constant spinor fields on $\mathcal{N}$ such that

$$P^{AA'}\lambda^0_A\lambda^0_{A'} \geqslant I(r_0) \tag{8}$$

for all constant spinors λ^0_A. Secondly, we use the Witten equation on the compact space-like hypersurface $\mathcal{S}$ to show that

$$I(r_0) \geqslant 0$$

and hence

$$P^{AA'}\lambda^0_A\lambda^0_{A'} \geqslant 0.$$

Since λ^0_A is arbitrary, this implies that $P^{AA'}$ is future pointing.

To prove the first step we use the following propagation law on $\mathcal{N}$:

$$\partial\lambda_0/\partial r = 0, \tag{9}$$

$$\partial\lambda_1/\partial r - 2(\eth'\lambda_0 + \rho\lambda_1) = 0 \tag{10}$$

where $\rho = \iota^A o^{A'} o^B \nabla_{AA'} o_B$ is the divergence of $\mathcal{N}$ and $\eth$ and $\eth'$ are the GHP 'edth' operators which, in our case, are asymptotically related to the NP 'edth' operators by

$$\lim_{r\to\infty}(r\eth) = -\eth_0, \qquad \lim_{r\to\infty}(r\eth') = -\bar{\eth}_0. \tag{11}$$

It is clear from equations (9) and (10) that λ_0 and λ_1 are determined over the whole of $\mathcal{N}$ once they are specified on some $r=$constant cut of $\mathcal{N}$. Furthermore, on using the asymptotic equation

$$\rho = -r^{-1} + \mathrm{O}(r^{-2}) \tag{12}$$

together with equations (11), it can be seen that equations (1) and (3) automatically hold for such a spinor field. Thus, since λ_0 is independent of r by equation (9), our propagation law produces an asymptotically constant spinor field on $\mathcal{N}$ once λ_0 and λ_1 are specified on $r=r_0$, with λ_1 arbitrary and λ_0 satisfying equation (2).

After a relatively easy spin coefficient calculation which uses equations (6), (7), (9) and (10), one can show that

$$\frac{\mathrm{d}I}{\mathrm{d}r} = \oint (X\bar{X} + 3Y\bar{Y} - \tfrac{1}{2}G_{ab}l^a k^b)\,\mathrm{d}\Omega \tag{13}$$

where

$$X = o^A \iota^{A'} o^B \nabla_{BA'}\lambda_A, \tag{14}$$

$$Y = o^A o^{A'} \iota^B \nabla_{BA'}\lambda_A, \tag{15}$$

$$k^a = \lambda^A \lambda^{A'}, \tag{16}$$

and G_{ab} is the Einstein tensor. By the dominant energy condition we have

$$-G_{ab}k^a l^b \geqslant 0$$

(the minus sign appears here due to our choice of sign in the definition of the Riemann tensor) and hence (13) gives

$$dI/dr \geqslant 0. \tag{17}$$

Combining these results, we see that the inequality (8) automatically holds for the spinor fields defined above.

To prove the second step we need only show that there exist spinor fields on the compact space-like hypersurface $\mathscr{L}$ which make $I(r_0)$ positive and for which the component λ_0 satisfies equation (2) on the boundary S_0. It is important to note that no restriction need be imposed on the component λ_1 on the boundary. From the results of our earlier paper (Ludvigsen and Vickers 1981) it is clear that any non-singular solution of the Witten equation on $\mathscr{N}$ guarantees that $I(r_0)$ is positive. Furthermore, since the Witten equation on a space-like hypersurface consists of an elliptic system of two first-order partial differential equations, standard existence theorems can be used to show that specifying λ_0 on S_0 uniquely determines a non-singular solution of these equations on $\mathscr{L}$. In particular we may choose λ_0 to satisfy equation (2). This leads to the inequalities

$$P^{AA'}\overset{0}{\lambda}_A\overset{0}{\lambda}_{A'} \geqslant I(r_0) \geqslant 0,$$

thus showing that $P^{AA'}$ is future pointing or, in other words, the Bondi mass is positive.

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